INFORMATION AND COMPUTATION **82**, 65–80 (1989)

Recursive Domain Equations for Concrete Data Structures

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In the theory of denotational semantics of programming languages, we study event structures which combine features of Scott's information systems and Kahn and Plotkin's concrete data structures and model computational processes. We show that a simple approximation concept for event structures allows us to obtain straightforward solutions of recursive domain equations for event domains. From this, we derive a generalization of corresponding theorems of Kahn and Plotkin respectively Berry and Curien for concrete data structures and concrete domains.

1. INTRODUCTION

In the theory of denotational semantics of programming languages, various kinds of systems of information and associated partial orders (domains) of information have been extensively studied. Scott (1982) considered information systems and domains. Kahn and Plotkin (1978) introduced concrete data structures and concrete domains. Winskel (1981, 1987) (cf. also Nielsen, Plotkin, and Winskel, 1981) studied a generalization, the event structures and event domains. In this paper, we wish to consider how to solve recursive domain equations for event domains and to generalize corresponding theorems of Kahn and Plotkin (1978) and Berry and Curien (1982) for distributive concrete domains.

Recursive domain equations are usually considered as fixpoint equations to be solved in categories instead of complete partial orders (cf. Lehmann and Smyth, 1977; Smyth and Plotkin, 1977; Wand, 1975). Domains may "approximate" each other in various ways, the classical and appropriate concept of approximation being that of embedding (Scott, 1971; Smyth and Plotkin, 1977; Stoy, 1977). Hence one applies a categorical version of the usual Knaster–Tarski theorem for cpo's and obtains solutions of domain equations only up to isomorphism. Here we will take up an approach of Berry and Curien (1982) for concrete data structures, which has also been applied recently to Scott's information systems by Larsen

0890-5401/89 $3.00

and Winskel (1984). See also Curien (1986) for further background information on this topic.

First let us introduce some notation. An *event structure* $\mathscr{E}$ consists of a set E of tokens together with a consistency relation for finite subsets of E and an enabling relation between finite subsets and elements of E satisfying certain natural axioms. The elements of E can be thought of, e.g., as the units of information which can in principle be computed by a machine, whereas the enabling relation describes the computation possibilities themselves. A *state of information* is a subset X of E such that each finite subset of X is consistent and each element of X can be deduced through finitely many successive applications of the enabling relation from a finite number of elements of X which are "a priori true," i.e., enabled by the empty set. The set of all such states of information, partially ordered under inclusion, is denoted by $(D(\mathscr{E}), \subseteq)$. An *event domain* is an axiomatically defined particular algebraic complete order $(D, \leqslant)$; these domains naturally generalize Kahn and Plotkin's concrete domains. In Droste (1989) we showed that for any event structure $\mathscr{E}$, $(D(\mathscr{E}), \subseteq)$ is an event domain; conversely, for any event domain $(D, \leqslant)$ there exists an event structure $\mathscr{E}$ which "generates" $(D, \leqslant)$, i.e., for which $(D(\mathscr{E}), \subseteq)$ is isomorphic to $(D, \leqslant)$. This generalizes results of Winskel (1981, 1987) who obtained the corresponding characterization theorems under the additional assumption that either $\mathscr{E}$ is stable or that the consistency relation is induced by a binary conflict relation on E (cf. Winskel, 1987, end of Section 1.1).

Now let $\mathscr{E}$, $\mathscr{E}'$ be two event structures with underlying sets E, E', respectively. As already in Larsen and Winskel (1984), Winskel (1987) (and as usual in model theory), we say that $\mathscr{E}$ is a substructure of $\mathscr{E}'$, denoted $\mathscr{E} \subseteq \mathscr{E}'$, if E is a subset of E' and the consistency and enabling relations on E are just the restrictions of the corresponding relations of $\mathscr{E}'$. Under this substructure relation, the class of all event structures becomes a complete partial order, on which the usual operations (e.g., taking products, sums) are continuous (cf. Winskel, 1987). We show for any two event domains $(D, \leqslant)$, $(D', \leqslant)$ that there exists a nice stable injection–projection pair from D to D', iff D, D' are generated by event structures $\mathscr{E}$, $\mathscr{E}'$, respectively, such that $\mathscr{E} \subseteq \mathscr{E}'$.

This result allows us to solve fixpoint equations for event domains in the complete partial order of the more concrete event structures and thus to obtain *exact* solutions, not just isomorphisms. In particular, we show that Berry and Curien's corresponding result holds also for non-distributive concrete domains with an analogously defined substructure relation (which is stronger than their concept of "inclusion") for concrete data structures. In general, our constructions differ from the ones of Berry and Curien, but in the distributive case they turn out to be equivalent.

The organization of this paper is as follows. In Section 2 we briefly review the basic properties of event structures and event domains from Droste (1989). In Section 3 we prove our main result mentioned above. Finally, in Section 4 we apply our result to concrete domains and concrete data structures.

2. Event Structures and Event Domains

In this section, we study the basic properties of event structures and event domains, referring the reader for details to Droste (1989), and Winskel (1987). For any set E, let $\mathrm{Fin}(E)$ be the system of all finite subsets of E.

Definition 2.1 (cf. Winskel, 1987). An *event structure* is a triple $\mathscr{E} = (E, \mathrm{Cons}, \vdash)$ satisfying the conditions:

(a) E is a set (the units of information);

(b) $\mathrm{Cons} \subseteq \mathrm{Fin}(E)$ is non-empty (the *consistent* sets) and whenever $A \subseteq B$ and $B \in \mathrm{Cons}$, then $A \in \mathrm{Cons}$;

(c) $\vdash \subseteq \mathrm{Cons} \times E$ (the *enabling relation* between consistent subsets and elements of E) and whenever $A \vdash e$, $A \subseteq B$, and $B \in \mathrm{Cons}$, then $B \vdash e$.

If there is no ambiguity, we also denote $(E, \mathrm{Cons}, \vdash)$ simply by E. We say that a set $A \subseteq E$ is *consistent* iff $A \in \mathrm{Cons}$. A subset X of E is a *state* of E, if two conditions are satisfied:

(1) $A \subseteq X$, A finite $\Rightarrow A \in \mathrm{Cons}$ (consistency)

(2) $e \in X \Rightarrow \exists e_1, ..., e_n \in X$ such that $e_n = e$ and

$$\forall i \leqslant n, \qquad \{e_j : j < i\} \vdash e_i \qquad \text{(deductibility)}.$$

The set of all states of E, partially ordered by inclusion, is denoted by $(D(\mathscr{E}), \subseteq)$ (or simply $(D(E), \subseteq)$) and called the *canonical event domain associated with E.*

Next we wish to characterize the partial orders $(D, \leqslant)$ occuring as canonical event domains $(D(E), \subseteq)$. Our notation needed for this task is standard (cf. Curien, 1986); we summarize it here for the convenience of the reader.

Let $(D, \leqslant)$ be a partially ordered set. For $x, y \in D$ we write $x \uparrow y$ if there is $z \in D$ with $x \leqslant z$ and $y \leqslant z$, and $x \not\uparrow y$ otherwise. A non-empty subset A of D is *directed* if for any $a, b \in A$ there is $c \in A$ with $a \leqslant c$ and $b \leqslant c$. $(D, \leqslant)$ is *complete*, if D has a smallest element, denoted by $\perp$, and any directed subset of D has a supremum in D. An element $x \in D$ is called *isolated* (or *compact*), if for any directed subset A of D for which $\sup A$ exists and

$x \leqslant \sup A$ there is $y \in A$ with $x \leqslant y$. The set of all isolated points of D is denoted by D^0. Then D is *algebraic*, if for each $x \in D$ the set $\{d \in D^0: d \leqslant x\}$ is directed and has x as supremum. Let $x, y \in D$. We write $x \prec y$ if y covers x, i.e., if $x < y$ and there is no $z \in D$ with $x < z < y$. A *chain* from x to y is a sequence $x_0, ..., x_n$ in D such that $x = x_0$, $y = x_n$, and $x_i \prec x_{i+1}$ for each $i = 0, ..., n-1$. A *prime interval* of D is a pair (x, x') such that $x, x' \in D^0$ and $x \prec x'$; this pair is then denoted by $[x, x']$. For prime intervals we put $[x, x'] \prec [y, y']$ if $x \prec y$, $x' \prec y'$, and $y \neq x'$. If s, t are prime intervals, a *zigzag from s to t* is a sequence $s_0, ..., s_n$ of prime intervals of D such that $s = s_0$, $t = s_n$, and for all $0 \leqslant i < n$ either $s_i \prec s_{i+1}$ or $s_{i+1} \prec s_i$. We call two prime intervals $[x, x']$ and $[y, y']$ *equivalent*, denoted by $[x, x'] \asymp [y, y']$, if there exists a zigzag from $[x, x']$ to $[y, y']$. The equivalence class of $[x, x']$ is denoted by $[x, x']_{\asymp}$. For any $x \in D$, we put $s(x) = \{[z, z']_{\asymp} : z' \leqslant x\}$. Clearly $x \leqslant y$ implies $s(x) \subseteq s(y)$. Now we can state our formal definition of an event domain:

DEFINITION 2.2. An *event domain* is an algebraic complete partial order $(D, \leqslant)$ satisfying the following conditions for any $x, x', y, y', z \in D^0$:

(F) $\{d \in D: d \leqslant x\}$ is finite;

(C) if $x \prec y$, $x \prec z$, $y \neq z$, and $y \uparrow z$, then $y \vee z$ exists and $y \prec y \vee z$, $z \prec y \vee z$;

(I) $[x, x'] \asymp [y, y']$ and $x \leqslant y$ imply $x' \leqslant y'$.

We say that an event structure $\mathscr{E}$ *generates* an event domain $(D, \leqslant)$, if $(D, \leqslant)$ and $(D(\mathscr{E}), \subseteq)$ are order-isomorphic.

As shown in Droste (1989), this concept of event domains includes (and generalizes) the one given in Curien (1986, Section 2.2). A partially ordered set $(P, \leqslant)$ is called *Dedekind-complete*, if any non-empty subset S of P which is bounded above in P has a supremum in $(P, \leqslant)$; equivalently, any non-empty subset of P which is bounded below in P has an infimum in $(P, \leqslant)$.

PROPOSITION 2.3 (Droste, 1989). *Let E be an event structure. Then $(D(E), \subseteq)$ is a Dedekind-complete event domain whose isolated elements are precisely the finite states of E and in which suprema are unions.*

Next we associate with each event domain a canonical event structure.

DEFINITION 2.4. Let $(D, \leqslant)$ be an event domain. We define an event structure $\mathscr{E}_D = (E_D, \mathrm{Cons}, \vdash)$ as follows:

(1) Let E_D be the set of all equivalence classes of prime intervals of D.

(2) Let Cons be the system of all finite subsets A of E_D such that $A = \{[a_i, a_i']_{\succ\!\prec} : i \in I\}$ and the set $\{a_i' : i \in I\}$ is bounded above in D.

(3) If $A \in \mathrm{Cons}$ and $e \in E_D$, put $A \vdash e$ iff $e = [x, x']_{\succ\!\prec}$ and $s(x) \subseteq A$ for some $x, x' \in D^0$.

Then $\mathscr{E}_D = (E_D, \mathrm{Cons}, \vdash)$ is called the *canonical event structure associated with* $(D, \leqslant)$.

We note that if $x \in D^0$ and $(x_i)_{i \leqslant n}$ is any chain from $\perp$ to x, then $s(x) = \{[x_i, x_{i+1}]_{\succ\!\prec} : 0 \leqslant i < n\}$. In particular, $s(x)$ is finite. This shows that in $(E_D, \mathrm{Cons}, \vdash)$, $s(x) \vdash [x, x']_{\succ\!\prec}$ for any $x, x' \in D^0$ with $x \prec x'$. Now we give the representation theorem for event domains.

THEOREM 2.5 (Droste, 1989, Theorem 2.7). *Let $(D, \leqslant)$ be an event domain and $(E_D, \mathrm{Cons}, \vdash)$ the canonical event structure associated with $(D, \leqslant)$. Then the mapping*

$$s\colon (D, \leqslant) \to (D(E_D), \subseteq), \text{ defined by } x \mapsto s(x) \qquad (x \in D),$$

is an isomorphism.

Next we recall the substructure relationship for event structures from Larsen and Winskel (1984) and Winskel (1987).

DEFINITION 2.6. Let $\mathscr{E} = (E, \mathrm{Cons}, \vdash)$ and $\mathscr{E}' = (E', \mathrm{Cons}', \vdash')$ be two event structures. Then $\mathscr{E}$ is a *substructure* of $\mathscr{E}'$, denoted $\mathscr{E} \subseteq \mathscr{E}'$, if the following conditions hold:

(1) $E \subseteq E'$;

(2) $\mathrm{Cons} = \{A \subseteq E : A \in \mathrm{Cons}'\}$;

(3) whenever $A \subseteq E$ is finite and $x \in E$, then $A \vdash x$ iff $A \vdash' x$.

Note that this concept coincides precisely with the usual modeltheoretic usage of the term "substructure" (for this, interpret Cons and $\vdash$ in the canonical way as sequences of finitary relations on E).

In the following let EVENT denote the class of all event structures. Then (EVENT, $\subseteq$), where $\subseteq$ is the substructure relation defined above, satisfies all axioms of a partial ordering except that EVENT is a class, not a set. Nevertheless we will simply say that (EVENT, $\subseteq$) is a partial order, and similarly all of the subsequent statements about EVENT are to be interpreted. The following is immediate:

PROPOSITION 2.7. *The class* (EVENT, $\subseteq$) *is an algebraic complete partial order. The isolated elements are those event structures* $(E, \mathrm{Cons}, \vdash)$ *for which E is finite. Suprema of subsets of* EVENT, *if existent, are obtained by taking componentwise set unions.*

Winskel (1987) showed that the usual operations (e.g., product, sum) on stable event structures are continuous. We just note here that this carries over with the same arguments to arbitrary event structures. We refer the reader to Winskel (1987) for a further discussion of categorical constructions on event structures.

3. Solving Recursive Domain Equations

In this section we will characterize when two event domains $(D, \leqslant)$, $(D', \leqslant)$ can be generated by event structures $\mathscr{E}$, $\mathscr{E}'$, respectively, such that $\mathscr{E} \subseteq \mathscr{E}'$. As mentioned in the Introduction, this allows us, in view of Proposition 2.7, to use the ordinary Knaster–Tarski theorem for complete partial orders to solve recursive domain equations for event domains in (EVENT, $\subseteq$) and thus obtain exact solutions, not just isomorphisms. We will need the notions of nice ideals and of retractions or stable injection–projection pairs between event domains. The latter were introduced by Kahn and Plotkin (1978); see also Berry and Curien (1982), Curien (1986), and Winskel (1987).

Definition 3.1. Let $(P, \leqslant)$ be a partially ordered set. A non-empty subset S of P is called an *ideal*, if the following two conditions are satisfied:

(1) Whenever $x \in P$, $s \in S$, and $x \leqslant s$, then $x \in S$.

(2) If $x, y \in S$ and $z \in P$ satisfy $z = x \vee y$ in $(P, \leqslant)$, then $z \in S$.

An ideal S of $(P, \leqslant)$ is *closed* in P, if it satisfies:

(3) If $A \subseteq S$ and $z \in P$ with $z = \sup A$ in $(P, \leqslant)$, then $z \in S$.

An ideal S of $(P, \leqslant)$ is called *nice* in $(P, \leqslant)$, if it satisfies:

(4) Whenever $x, x', y \in S$ and $y' \in P$ such that $x \prec x'$, $y \prec y'$, and $[x, x'] \rightarrowtail\!\!\!\prec [y, y']$ in $(P, \leqslant)$, then $y' \in S$.

First we show:

Proposition 3.2. *Let* $\mathscr{E} = (E, \mathrm{Cons}, \vdash)$ *and* $\mathscr{E}' = (E', \mathrm{Cons}', \vdash')$ *be two event structures such that* $\mathscr{E} \subseteq \mathscr{E}'$. *Then* $D(\mathscr{E}) = \{A \subseteq E: A \in D(\mathscr{E}')\}$. *In particular,* $D(\mathscr{E})$ *is a nice closed ideal of* $D(\mathscr{E}')$.

Proof. The first assertion is immediate by checking the definitions. To prove that $D(\mathscr{E})$ is closed in $D(\mathscr{E}')$, let $x_i \in D(\mathscr{E})$ $(i \in I)$ and $x \in D(\mathscr{E}')$ with $x = \sup\{x_i : i \in I\}$ in $(D(\mathscr{E}'), \subseteq)$. Then $x = \bigcup_{i \in I} x_i \subseteq E$ and thus $x \in D(\mathscr{E})$. To show that $D(\mathscr{E})$ is nice in $D(\mathscr{E}')$, let $x, x', y \in D(\mathscr{E})$ and $y' \in D(\mathscr{E}')$ with $x \prec x'$, $y \prec y'$, and $[x, x'] \rightarrowtail\!\!\!\prec [y, y']$ in $(D(\mathscr{E}'), \subseteq)$. Then $x' = x \cup \{e\}$, $y' = y \cup \{e\}$ for some $e \in E'$. As $x' \subseteq E$, we have $e \in E$ and thus $y' \in D(\mathscr{E})$.

Next we wish to prove a converse of Proposition 3.2. Let $(D, \leqslant)$ be an event domain and $\mathscr{E} = (E_D, \mathrm{Cons}, \vdash)$ the canonical event structure associated with $(D, \leqslant)$. We say that an event structure $\mathscr{E}^* = (E_D, \mathrm{Cons}^*, \vdash)$ is *associated with* $(D, \leqslant)$, if $D(\mathscr{E}^*) = D(\mathscr{E})$. Clearly, by Theorem 2.5 then $s: (D, \leqslant) \to (D(\mathscr{E}^*), \subseteq)$ is an isomorphism, and in particular $\mathscr{E}^*$ generates $(D, \leqslant)$. Note that $\mathscr{E}^*$ may differ from $\mathscr{E}$ (only) in its set of consistent sets; this will enable us below to construct an event structure which generates a given domain and which, at the same time, is a substructure of another event structure.

DEFINITION 3.3. Let $(D, \leqslant)$ be an event domain, S a nice ideal of D, and $\mathscr{E} = (E_D, \mathrm{Cons}, \vdash)$ an event structure associated with $(D, \leqslant)$. We put

$$E_D(S) = \{[x, x']_{\succ\!\prec} : x, x' \in S\},$$

$$\mathrm{Cons}^* = \mathrm{Cons} \cap \mathrm{Fin}(E_D(S)),$$

and for each set $A \in \mathrm{Cons}^*$ and $e \in E_D(S)$ let

$$A \vdash^* e \quad \text{iff} \quad e = [x, x']_{\succ\!\prec} \quad \text{for some} \quad x, x' \in S \quad \text{such that} \quad s(x) \subseteq A.$$

Then $\mathscr{E}^* = (E_D(S), \mathrm{Cons}^*, \vdash^*)$ is an event structure, called the *canonical event structure associated in* $\mathscr{E}$ *with* S.

Note here that $E_D(S)$ will not be E_S, because the equivalence classes are taken in D and not in S. Next we wish to show that under the assumptions of Definition 3.3, $\mathscr{E}^*$ is a substructure of $\mathscr{E}$. The following technical result will be useful.

LEMMA 3.4. *Let* $(D, \leqslant)$ *be an event domain and* S *a nice ideal of* D. *If* $y \in D^0$ *satisfies* $s(y) \subseteq E_D(S)$, *then* $y \in S$.

Proof. Let $(y_i)_{i \leqslant n}$ be a chain from $\perp$ to y. For each $i \leqslant n$, we have $[y_i, y_{i+1}] \succ\!\prec [x_i, x_i']$ for some $x_i, x_i' \in S$. Hence $y_i \in S$ implies $y_{i+1} \in S$. As $\perp \in S$, we get $y \in S$.

Now we can prove the converse of Proposition 3.2.

PROPOSITION 3.5. *Under the assumptions of Definition* 3.3, *we have* $\mathscr{E}^* \subseteq \mathscr{E}$. *Moreover, if* S *is a closed ideal of* D, *then the mapping*

$$s: (S, \leqslant) \to (D(\mathscr{E}^*), \subseteq), \qquad \textit{defined by} \quad x \mapsto s(x) \ (x \in S),$$

is an isomorphism. In particular, $\mathscr{E}^*$ *generates* $(S, \leqslant)$.

Proof. Clearly $E_D(S) \subseteq E_D$. Now let $A \subseteq E_D(S)$ be finite and $e \in E_D(S)$. Trivially, $A \vdash^* e$ implies $A \vdash e$. Now assume $A \vdash e$. Then $e = [y, y']_{\succ\!\prec}$

for some y, $y' \in D^0$ with $s(y) \subseteq A \subseteq E_D(S)$. Thus y, $y' \in S$ by Lemma 3.4, showing $A \vdash\!\!-^* e$. Hence $\mathscr{E}^* \subseteq \mathscr{E}$.

Now let S be a closed ideal of D. By Proposition 3.2, $D(\mathscr{E}^*) = \{A \subseteq E_D(S) \colon A \in D(\mathscr{E})\}$ and $D(\mathscr{E}^*)$ is a closed ideal of $D(\mathscr{E})$. Note that $(S, \leqslant)$ and $(D(\mathscr{E}^*), \subseteq)$ are algebraic complete partial orders. Clearly the mapping s is well defined. Let $A \in D^0(\mathscr{E}^*)$. Then $A \in D^0(\mathscr{E})$, and by Theorem 2.5, $A = s(x)$ for some $x \in D^0$. But then $x \in S$ by Lemma 3.4. Again by Theorem 2.5, this shows that s maps $(S^0, \leqslant)$ isomorphically onto $(D^0(\mathscr{E}^*), \subseteq)$. As $s(x) = \bigcup \{s(x^0) \colon x^0 \in S^0,\ x^0 \leqslant x\}$ for any $x \in S$, we obtain that s is an isomorphism as claimed.

Next we wish to relate closed ideals of partially ordered sets with stable injection–projection pairs defined as follows.

Definition 3.6 (cf. Curien, 1986). Let $(P, \leqslant)$, $(Q, \leqslant)$ be two partially ordered sets and $\varphi \colon P \to Q$, $\psi \colon Q \to P$ two monotonic functions. We say that (φ, ψ) is a *stable injection–projection pair* or that $(\varphi, \psi) \colon (P, \leqslant) \to (Q, \leqslant)$ is a *sipp*, if the following conditions are satisfied:

(1) $\psi \circ \varphi = \mathrm{id}_P$

(2) if $x, y \in Q$ and $x \leqslant y$, then $(\varphi \circ \psi)(x) = x \wedge (\varphi \circ \psi)(y)$ in $(Q, \leqslant)$.

The following lemma is basically contained in Berry and Curien (1982) and partially in Kahn and Plotkin (1978).

Lemma 3.7. *Let $(P, \leqslant)$ be a Dedekind-complete partially ordered set with a smallest element.*

(a) *Let S be a closed ideal of P. Let $\varphi = \mathrm{id}_S$, and $\psi \colon P \to S$ such that $\psi(x) = \sup\{s \in S \colon s \leqslant x\}$ $(x \in P)$. Then (φ, ψ) is a sipp from $(S, \leqslant)$ into $(P, \leqslant)$.*

(b) *Let $(Q, \leqslant)$ be another partially ordered set and $(\varphi, \psi) \colon (P, \leqslant) \to (Q, \leqslant)$ be a sipp. Then $\varphi(P)$ is a closed ideal of Q, and φ is an isomorphism from $(P, \leqslant)$ onto $(\varphi(P), \leqslant)$.*

Proof. Argue as in Curien (1986, proof of Proposition 2.3.8).

We will call a sipp (φ, ψ) from $(P, \leqslant)$ into $(Q, \leqslant)$ *nice*, if $\varphi(P)$ is nice in Q. Now we can summarize our results.

Theorem 3.8. *Let $(D, \leqslant)$ and $(D', \leqslant)$ be two event domains. The following are equivalent:*

(1) *There exists a nice sipp from $(D, \leqslant)$ to $(D', \leqslant)$.*

(2) *$(D, \leqslant)$ is isomorphic to a nice closed ideal of $(D', \leqslant)$.*

(3) *For any event structure $\mathscr{E}'$ associated with $(D', \leqslant)$ there exists an event structure $\mathscr{E} \subseteq \mathscr{E}'$ which generates $(D, \leqslant)$.*

(4) *There are two event structures $\mathscr{E}$, $\mathscr{E}'$ generating D, D', respectively, such that $\mathscr{E} \subseteq \mathscr{E}'$.*

Proof. Note that $(D, \leqslant)$ is Dedekind-complete by Theorem 2.5 and Proposition 2.3 (or by Droste, 1989, Lemma 2.4(c)).

$(1) \to (2)$ Apply Lemma 3.7(b).

$(2) \to (3)$ Let S be the nice closed ideal of $(D', \leqslant)$ isomorphic to $(D, \leqslant)$, and let $\mathscr{E}$ be the canonical event structure associated in $\mathscr{E}'$ with S. Proposition 3.5 implies the result.

$(3) \to (4)$ Let $\mathscr{E}'$ be the canonical event structure associated with $(D', \leqslant)$, and apply (3).

$(4) \to (1)$ By Proposition 3.2, $D(\mathscr{E})$ is a nice closed ideal of $(D(\mathscr{E}'), \subseteq)$, and by Lemma 3.7(a) there exists a nice sipp from $(D, \leqslant) \cong (D(\mathscr{E}), \subseteq)$ to $(D(\mathscr{E}'), \subseteq) \cong (D', \leqslant)$.

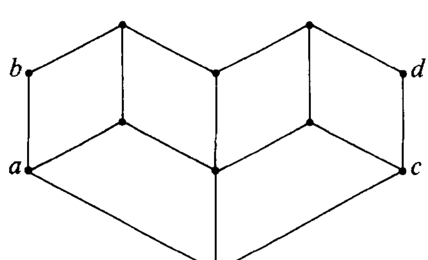
The above proof shows that in condition (4) of Theorem 3.8 we can take as $\mathscr{E}'$ the canonical event structure associated with $(D', \leqslant)$ and as $\mathscr{E}$ the canonical event structure associated in $\mathscr{E}'$ with a suitable closed ideal of $(D', \leqslant)$. In the remainder of this section we wish to study when E can be taken as (an isomorphic copy of) the canonical event structure associated with $(D, \leqslant)$.

Let us say that two event structures $\mathscr{E} = (E, \mathrm{Cons}, \vdash)$ and $\mathscr{E}' = (E', \mathrm{Cons}', \vdash')$ are *isomorphic*, if there exists a bijective mapping f from E onto E' such that:

(1) $A \in \mathrm{Cons}$ iff $f(A) \in \mathrm{Cons}'$ for any $A \subseteq E$;

(2) $A \vdash x$ iff $f(A) \vdash' f(x)$ for any $A \subseteq E$, $x \in E$.

First we give an example which shows that, in general, in condition (4) of Theorem 3.8, $\mathscr{E}$ is not isomorphic to the canonical event structure associated with $(D, \leqslant)$. Let $(D', \leqslant)$ be the following event domain having precisely 11 elements.



$(D', \leqslant)$

Let $D = \{\perp, a, b, c, d\}$. Then D is a closed ideal of $(D', \leqslant)$. We have $|E_{D'}| = |E_D| = 4$. Hence, if $\mathscr{E}_D$ were isomorphic to a substructure of $\mathscr{E}_{D'}$, we would obtain $\mathscr{E}_D \cong \mathscr{E}_{D'}$, and thus $(D, \leqslant) \cong (D(\mathscr{E}_D), \subseteq) \cong (D(\mathscr{E}_{D'}), \subseteq) \cong (D', \leqslant)$, a contradiction. We note that D' also contains a closed ideal S which is not nice: Take $S = \{\perp, a, b, c\}$.

Next we introduce conditions which are sufficient to exclude examples like the above. This is motivated by Curien (1986, Section 2).

DEFINITION 3.9 (Winskel, 1987). (a) An event structure $(E, \mathrm{Cons}, \vdash)$ is called *stable*, if whenever X is a state of E, $x \in X$, and $A, B \subseteq X$ are both minimal with respect to $A \vdash x$, $B \vdash x$, then $A = B$.

(b) An event domain $(D, \leqslant)$ is called *distributive*, if whenever $x, y, z \in D$ with $y \uparrow z$, then $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Let $(E, \mathrm{Cons}, \vdash)$ be an event structure. Assume $e, e_1, ..., e_n \in E$ are such that $e_n = e$ and

$$\forall i \leqslant n, \qquad \{e_j : j < i\} \vdash e_i. \tag{*}$$

Then we will call $\{e_1, ..., e_n\}$ a *deduction of* e.

LEMMA 3.10. *Let $(E, \mathrm{Cons}, \vdash)$ be a stable event structure. Assume X is a state of E, $x \in X$, and $A, B \subseteq X$ are two minimal deductions of x. Then $A = B$.*

Proof. Assume $A = \{x_1, ..., x_n\}$ and $B = \{y_1, ..., y_m\}$ such that $x_n = y_m = x$ and the enumeration is as above in (*). Choose $A' \subseteq A$, $B' \subseteq B$ minimal with respect to $A' \vdash x$, $B' \vdash x$. Then $A' = B'$ by stability of E and $x_{n-1} \in A'$, $y_{m-1} \in B'$ by minimality of A and B. Repeating this argument with x_n replaced by x_{n-1}, we obtain finally $A \subseteq B$ and, by a symmetrical argument, $B \subseteq A$. Hence $A = B$.

Arguing as in the proof of Curien (1986, Lemma 2.2.14), we obtain:

LEMMA 3.11. *Let $(D, \leqslant)$ be a distributive event domain and e any equivalence class of prime intervals of $(D, \leqslant)$. Then there exists a "minimum" representative $[x, x'] \in e$, i.e., $[x, x'] \in e$ and $x \leqslant y$ for any $[y, y'] \in e$.*

PROPOSITION 3.12. (a) *Let $\mathscr{E}$ be a stable event structure. Then $(D(\mathscr{E}), \subseteq)$ is distributive.*

(b) *Let $(D, \leqslant)$ be a distributive event domain and $\mathscr{E} = (E_D, \mathrm{Cons}, \vdash)$ any event structure associated with $(D, \leqslant)$. Then $\mathscr{E}$ is stable.*

(c) *Let $(D, <)$ be a distributive event domain and S an ideal of $(D, <)$. Then S is nice in $(D, \leqslant)$.*

Proof. (a) Note that for any states X, Y of $\mathscr{E}$ with $X \uparrow Y$ in $(D(\mathscr{E}), \subseteq)$ we have $X \wedge Y = X \cap Y$ by Lemma 3.10. The result follows.

(b) Let $e \in E_D$ and $A \subseteq E_D$ be minimal with $A \vdash e$. By Lemma 3.11, choose $[x, x'] \in e$ such that $x \leqslant y$ for any $[y, y'] \in e$. Then $s(x) \vdash e$ and $s(x) \subseteq s(y)$ for any $[y, y'] \in e$. Hence $A = s(x)$, and $\mathscr{E}$ is stable.

(c) Let x, x', $y \in S$ and $y' \in D$ with $x \prec x'$, $y \prec y'$, and $[x, x'] \asymp [y, y']$ in $(D, \leqslant)$. By Lemma 3.11, choose z, $z' \in D$ such that $[x, x'] \asymp [z, z'] \asymp [y, y']$, $z \leqslant x$, and $z \leqslant y$. Then $z' \leqslant x'$, $z' \leqslant y'$, and $y' = z' \vee y$. Hence z', $y' \in S$.

As an immediate consequence of Proposition 3.12 and Theorem 2.5, an event domain is distributive iff it is generated by a stable event structure.

PROPOSITION 3.13. *Let $(D, \leqslant)$ be a distributive event domain and S a closed ideal of D. Then $(S, \leqslant)$ is an event domain. Let $\mathscr{E}$ be the canonical event structure associated with $(D, \leqslant)$ and $\mathscr{E}^*$ the canonical event structure associated in $\mathscr{E}$ with S. Then $\mathscr{E}^* = \mathscr{E}_S$, the canonical event structure associated with $(S, \leqslant)$.*

Proof. An argument as in Curien (1986, p. 146) shows that if $x, x', y, y' \in S$ with $x \prec x'$, $y \prec y'$, and $[x, x'] \asymp [y, y']$ in $(D, \leqslant)$, then $[x, x'] \asymp [y, y']$ also in $(S, \leqslant)$, and that $(S, \leqslant)$ is an event domain. Thus, if we identify each equivalence class in S of prime intervals in S with the corresponding equivalence class in D, we obtain $E_D(S) = E_S$ (up to a canonical bijection). Now let $A \subseteq E_D(S)$ be finite and consistent in $\mathscr{E}^*$. Then there is $z \in D$ such that $A = \{[y_i, y_i']_{\asymp} : i \in I\}$ with $y_i, y_i' \in D$ and $y_i' \leqslant z$ for all $i \in I$ (I finite). By Lemma 3.11 choose $x_i, x_i' \in D$ such that $[x_i, x_i']$ is a minimum representative of $[y_i, y_i']_{\asymp}$; then $x_i' \leqslant y_i' \leqslant z$ and $x_i, x_i' \in S$ by $A \subseteq E_D(S)$ for all $i \in I$. Hence $\{x_i' : i \in I\}$ has an upper bound in S, showing that A is consistent in $\mathscr{E}_S$. Thus $\mathscr{E}^* = \mathscr{E}_S$.

Let us call an event structure $\mathscr{E}$ a *canonical event structure*, if for some event domain $(D, \leqslant)$, $\mathscr{E}$ is isomorphic to $\mathscr{E}_D$, the canonical event structure associated with $(D, \leqslant)$. In Droste (1989) we gave an axiomatic description of canonical event structures and derived further properties. Now we can answer our initial question.

THEOREM 3.14. *Let $(D, \leqslant)$ and $(D', \leqslant)$ be two event domains such that $(D', \leqslant)$ is distributive. The following are equivalent*:

(1) *There exists a sipp from $(D, \leqslant)$ to $(D', \leqslant)$.*

(2) *There are two stable canonical event structures $\mathscr{E}, \mathscr{E}'$ generating D, D', respectively, such that $\mathscr{E} \subseteq \mathscr{E}'$.*

Proof. (1) → (2) By Lemma 3.7(b) and Proposition 3.12(c), $(D, \leqslant)$ is isomorphic to a nice closed ideal S of $(D', \leqslant)$. Clearly $(S, \leqslant)$ is distributive. By Theorem 2.5, Proposition 3.13, and Proposition 3.5 there are two canonical event structures $\mathscr{E}, \mathscr{E}'$ generating D, D', respectively, such that $\mathscr{E} \subseteq \mathscr{E}'$. By Proposition 3.12(b), $\mathscr{E}$ and $\mathscr{E}'$ are stable.

(2) → (1) Immediate by Theorem 3.8.

4. Concrete Data Structures and Concrete Domains

In this section we will apply Theorem 3.8 to the concrete data structures and concrete domains of Kahn and Plotkin (1978) in order to generalize a classical result of Berry and Curien (1982, Theorem 6.2.7). First let us apply Theorem 3.8 to particular event structures and event domains of Winskel (1981) (cf. Curien, 1986). The subsequent definition of conflict event structures, in which a symmetric binary relation of conflict is replaced by a consistency predicate, can be easily seen to be equivalent to Winskel's original one.

Definition 4.1. (a) An event structure $(E, \mathrm{Cons}, \vdash)$ is called a *conflict event structure*, if for any finite subset A of E, A is consistent iff each subset B of A with precisely two elements is consistent.

(b) A *conflict event domain* is an algebraic complete partial order $(D, \leqslant)$ satisfying conditions (F), (C) (cf. Definition 2.2) and for any $x, x', x'', y, y', y'' \in D^0$ the following two axioms:

(R) $[x, x'] \rtimes\!\!\!\ltimes [x, x'']$ implies $x' = x''$;

(V) $[x, x'] \rtimes\!\!\!\ltimes [y, y']$, $[x, x''] \rtimes\!\!\!\ltimes [y, y'']$, and $x' \uparrow x''$ imply $y' \uparrow y''$.

As is easy to see, for any conflict event structure $(E, \mathrm{Cons}, \vdash)$, $(D(E), \subseteq)$ is a conflict event domain. Conversely, as shown in Droste (1989, Proposition 2.19), any conflict event domain $(D, \leqslant)$ is an event domain and can hence be generated by an event structure (Theorem 2.5). Moreover, we have:

Proposition 4.2 (Winskel, 1981; cf. Droste, 1989, Corollary 2.10). *Let $(D, \leqslant)$ be a conflict event domain and $\mathscr{E} = (E_D, \mathrm{Cons}, \vdash)$ the canonical event structure associated with $(D, \leqslant)$. Let* Cons* *be the system of all finite subsets A of E_D such that whenever $[z, z']_{\rtimes\!\ltimes}$, $[z, z'']_{\rtimes\!\ltimes} \in A$, then $z' \uparrow z''$ in $(D, \leqslant)$. Then $\mathscr{E}^* = (E_D, \mathrm{Cons}^*, \vdash)$ is a conflict event structure satisfying $D(\mathscr{E}^*) = D(\mathscr{E})$.*

Now we obtain the following version of Theorem 3.8 for conflict event structures and conflict event domains:

COROLLARY 4.3. *Let* $(D, \leqslant)$ *and* $(D', \leqslant)$ *be two conflict event domains. The following are equivalent*:

(1) *There exists a nice sipp from* $(D, \leqslant)$ *to* $(D', \leqslant)$.

(2) *For any conflict event structure* $\mathscr{E}'$ *associated with* $(D', \leqslant)$ *there exists a conflict event structure* $\mathscr{E} \subseteq \mathscr{E}'$ *which generates* $(D, \leqslant)$.

(3) *There are two conflict event structures* $\mathscr{E}, \mathscr{E}'$ *generating* D, D', *respectively, such that* $\mathscr{E} \subseteq \mathscr{E}'$.

Proof. $(1) \to (2)$ By Theorem 3.8, there is an event structure $\mathscr{E} \subseteq \mathscr{E}'$ which generates $(D, \leqslant)$. Clearly $\mathscr{E}$ is a conflict event structure.

$(2) \to (3)$ Apply Proposition 4.2 and (2).

$(3) \to (1)$ Immediate by Theorem 3.8.

Next we introduce the *concrete data structures* and *concrete domains* of Kahn and Plotkin (1978).

DEFINITION 4.4 (cf., e.g., Curien, 1986). (a) A *concrete data structure* is a quadruple $\mathscr{M} = (C, V, E, \vdash)$ such that

(1) C, V, E are sets (of *cells*, *values*, and *events*, respectively) with $E \subseteq C \times V$;

(2) $\forall c \in C, \exists v \in V.\ (c, v) \in E$ ("any cell may be filled");

(3) $\vdash \subseteq \mathrm{Fin}(E) \times C$, i.e., $\vdash$ is a relation (the *enabling* relation) between finite subsets of E and elements of C.

A subset X of E is called a *state* of $\mathscr{M}$, if the following two conditions are satisfied:

(i) $(c, v_1), (c, v_2) \in X \Rightarrow v_1 = v_2$ (consistency)

(ii) $\forall e \in X, \exists e_i = (c_i, v_i) \in X\ (i = 1, ..., n)$ such that $e_n = e$ and

$\forall i \leqslant n, \exists X_i \subseteq \{e_j : j < i\}.\ X_i \vdash c_i$ (deductibility).

The set of all states of $\mathscr{M}$, partially ordered by inclusion, is denoted by $(D(\mathscr{M}), \subseteq)$. If $(D, \leqslant)$ is a partially ordered set isomorphic to $(D(\mathscr{M}), \subseteq)$, we say that $\mathscr{M}$ *generates* $(D, \leqslant)$.

(b) Let $\mathscr{M} = (C, V, E, \vdash)$ and $\mathscr{M}' = (C', V', E', \vdash')$ be two concrete data structures. We say that $\mathscr{M}$ is a *substructure* of $\mathscr{M}'$, denoted $\mathscr{M} \subseteq \mathscr{M}'$, if the following conditions hold:

(1) $C \subseteq C'$, $V \subseteq V'$, and $E \subseteq E'$;

(2) whenever $A \subseteq E$ is finite and $c \in C$, then $A \vdash c$ iff $A \vdash' c$.

Note that the requirements for $\mathcal{M}$ to be a substructure of $\mathcal{M}'$ are stronger than the ones of Berry and Curien (1982) or Curien (1986) saying that $\mathcal{M}$ is "included" in $\mathcal{M}'$. For instance, if $\mathcal{M}, \mathcal{M}'$ have the same sets of cells, values, and events, respectively, and if $\mathcal{M} \subseteq \mathcal{M}'$ (in our sense), then $\mathcal{M} = \mathcal{M}'$. Let CDS denote the class of all concrete data structures. As for event structures, (CDS, $\subseteq$) is a complete partial order, and the usual operations on concrete data structures like product, separated sum, and exponential (cf. Berry and Curien, 1982; Curien, 1986), are continuous functions from $(\text{CDS}, \subseteq)^2$ into (CDS, $\subseteq$). For more examples of such constructors, see Berry (1981).

DEFINITION 4.5. An algebraic complete partial order $(D, \leqslant)$ is called a *concrete domain*, if $(D, \leqslant)$ satisfies conditions (F), (C), (R) (cf. Definitions 2.2 and 4.1) and the following axiom:

(Q) Whenever $x, y, z \in D^0$ such that $z \prec x$, $z < y$, and $x \uparrow y$, then there exists a unique element $x' \in D^0$ such that $z \prec x' \leqslant y$ and $x \uparrow x'$.

By a well-known result of Kahn and Plotkin (1978), the set $(D(\mathcal{M}), \subseteq)$ of all states of a concrete data structure $\mathcal{M}$ is a concrete domain, and, conversely, any concrete domain $(D, \leqslant)$ is of this form. Moreover, any concrete domain is a conflict event domain (cf. Curien, 1986, Proposition 2.2.11) and hence generated by a particular type of conflict event structure, which we now formally define.

DEFINITION 4.6. Let $\mathscr{E} = (E, \text{Cons}, \vdash)$ be a conflict event structure. Define a binary relation $\#$ on E by putting $e_1 \# e_2$ iff $e_1 \neq e_2$ and $\{e_1, e_2\} \notin \text{Cons}$. We say that $\mathscr{E}$ is a *concrete event system*, if the following two conditions hold:

(1) The reflexive closure of $\#$ is transitive.

(2) Whenever $X \subseteq E$ is finite and $e_1, e_2 \in E$ with $e_1 \# e_2$, then $X \vdash e_1$ iff $X \vdash e_2$.

It is known that the classes of partial orders generated by concrete data structures and by concrete event systems, respectively, coincide (Curien, 1986, p. 139).

LEMMA 4.7. *Let $(D, \leqslant)$ be a concrete domain. There exists a concrete event system $\mathscr{E}$ which is associated with $(D, \leqslant)$.*

Proof. As noted above, $(D, \leqslant)$ is an event domain. Let $\mathscr{E}$ be the conflict event structure defined in Proposition 4.2. The argument in Curien (1986, Proof of Theorem 2.2.12) shows that $\mathscr{E}$ is concrete.

Now we can show:

THEOREM 4.8. *Let* $(D, \leqslant)$, $(D', \leqslant)$ *be two concrete domains. The following are equivalent*:

(1) *There exists a nice sipp from* $(D, \leqslant)$ *to* $(D', \leqslant)$.

(2) *There are concrete event systems* $\mathscr{E}, \mathscr{E}'$ *generating* D, D', *respectively, such that* $\mathscr{E} \subseteq \mathscr{E}'$.

(3) *There are concrete data structures* $\mathscr{M}, \mathscr{M}'$ *generating* D, D', *respectively, such that* $\mathscr{M} \subseteq \mathscr{M}'$.

Proof. $(1) \rightarrow (2)$ By Lemma 4.7, there is a concrete event system $\mathscr{E}'$ associated with $(D', \leqslant)$. By Corollary 4.3, there exists an event structure $\mathscr{E} \subseteq \mathscr{E}'$ which generates $(D, \leqslant)$. Clearly $\mathscr{E}$ is concrete.

$(2) \rightarrow (1)$ Immediate by Theorem 3.8.

Our argument for the equivalence $(2) \leftrightarrow (3)$ uses ideas of Curien (1986, p. 139).

$(2) \rightarrow (3)$ Define $\#$ on E' as in Definition 4.6, and let $\#^*$ be the reflexive closure of $\#$. For each $e \in E'$ let $[e]$ be the $\#^*$-equivalence class of e in E'. We put $C' = \{[e]: e \in E'\}$, $V' = E'$, $\tilde{E}' = \{([e], e): e \in E'\}$, and $C = \{[e]: e \in E\}$, $V = E$, $\tilde{E} = \{([e], e): e \in E\}$. For each finite $X = \{([e_i], e_i)\colon i = 1, ..., n\} \subseteq \tilde{E}'$ and $c \in C'$ let $X \vdash c$ iff $c = [e]$ for some $e \in E'$ such that $\{e_i : i = 1, ..., n\} \vdash e$. Then let $\mathscr{M} = (C, V, \tilde{E}, \vdash)$ and $\mathscr{M}' = (C', V', \tilde{E}', \vdash)$. Clearly $\mathscr{M} \subseteq \mathscr{M}'$, $D(\mathscr{M}), \subseteq) \cong (D(\mathscr{E}), \subseteq)$, and $(D(\mathscr{M}'), \subseteq) \cong (D(\mathscr{E}'), \subseteq)$.

$(3) \rightarrow (2)$ We define a binary relation $\#$ on E' by putting $e_1 \# e_2$ iff there are $c \in C'$, $v_1, v_2 \in V'$ such that $e_1 = (c, v_1)$, $e_2 = (c, v_2)$, and $v_1 \neq v_2$. Let Cons′ (Cons) be the system of all finite subsets A of E' (E), respectively, such that $\neg(e_1 \# e_2)$ for any $e_1, e_2 \in A$. For each finite subset X of E' and $(c, v) \in E'$ let $X \vdash (c, v)$ iff $X \vdash c$. Put $\mathscr{E}' = (E', \mathrm{Cons}', \vdash)$ and $\mathscr{E} = (E, \mathrm{Cons}, \vdash)$. Clearly $\mathscr{E} \subseteq \mathscr{E}'$ and $D(\mathscr{E}') = D(\mathscr{M}')$, $D(\mathscr{E}) = D(\mathscr{M})$.

Note here that if $(D', \leqslant)$ is distributive, then, by Lemma 3.7(b) and Proposition 3.12(c), any sipp from $(D, \leqslant)$ into $(D', \leqslant)$ is nice. Hence the implication $(1) \rightarrow (3)$ of Theorem 4.8 generalizes (the essential part of) Theorem 6.2.7 of Berry and Curien (1982), where the result was proved with a weaker version of "substructure" (termed "inclusion") and under the additional assumption that $(D', \leqslant)$ is distributive.

RECEIVED September 14, 1987; ACCEPTED July 29, 1988

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